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Generalized Kähler geometry and manifest $\mathcal{N} = (2, 2)$ supersymmetric nonlinear sigma-models

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Abstract: Generalized complex geometry is a new mathematical framework that is useful for describing the target space of $\mathcal{N} = (2, 2)$ nonlinear sigma-models. The most direct relation is obtained at the $\mathcal{N} = (1, 1)$ level when the sigma model is formulated with an additional auxiliary spinorial field. We revive a formulation in terms of $\mathcal{N} = (2, 2)$ semi-(anti)chiral multiplets where such auxiliary fields are naturally present. The underlying generalized complex structures are shown to commute (unlike the corresponding ordinary complex structures) and describe a Generalized Kähler geometry. The metric, B-field and generalized complex structures are all determined in terms of a potential K.

Keywords: Global Symmetries, Superstrings and Heterotic Strings, Superstring Vacua.

Contents

1. Introduction

In this paper, we show how a sigma-model based on semi-chiral and semi-antichiral superfields realizes a new mathematical concept called a generalized complex structure (GCS).

The geometry of the target space of supersymmetric nonlinear sigma-models is restricted by the number of supersymmetries in the base as well as the type of background fields (metric g and $NS-NS$ two-form B) in the target space [1−4]. The situation in two dimensions is summarized in table 1.

Some intermediate $\mathcal{N}=(p,q)$ geometries are also partly classified [5, 6].

We focus on $\mathcal{N}=(2,2)$ models in a metric and a B-field background, and find a reinterpretation of the bihermitean geometry. This is motivated by some recent advances in mathematics, initiated by Hitchin to describe generalized Calabi-Yau manifolds, e.g., including an antisymmetric B-field [7]. This Generalized Complex Geometry (GCG) [8] includes bihermitean geometry, Kähler geometry, and symplectic geometry as special cases. Its fundamental object is a Generalized Complex Structure (GCS), which is a map of the sum of the tangent and cotangent spaces $T\oplus T^*$ to itself; the GCS squares to minus one and obeys an integrability condition stated in terms of the Courant bracket, a general-

Supersymmetry	(0,0) or (1,1)	(2,2)	(2,2)	(4,4)	(4,4)
Background	g, B	g	g, B	g	g, B
Geometry	Riemannian	Kähler	bihermitian	hyperkähler	bihypercomplex

Table 1: The geometries of sigma-models with different supersymmetries.

ization of the Lie bracket to $T \oplus T^*$. The GCG is very well suited to a description of $\mathcal{N} = (2,2)$ in a nontrivial general background for many reasons, one of which is that the automorphism group of the Courant bracket includes the b-transform, which is precisely the gauge transformation of the B-field.

The realization of GCG in supersymmetric nonlinear sigma-models has recently been investigated in [9], (a preliminary study of the $\mathcal{N} = (2,2)$ model) and in [10], where the relation to GCS was established for the $\mathcal{N} = (2,0)$ model. The full $\mathcal{N} = (2,2)$ model in terms of its most general $\mathcal{N} = (1,1)$ formulation is not yet fully understood. Here we investigate how the GCS arises in models based on semi-(anti)chiral multiplets [11]. These models have an underlying bihermitean geometry with noncommuting complex structures and a closed off-shell supersymmetry algebra. This sets them apart from the other known models with bihermitean geometry and off-shell closure, i.e. those formulated in terms of chiral and twisted chiral multiplets, which have complex structures that necessarily commute.

In [8] the concept of generalized Kähler geometry is introduced and shown to encode the bihermitean geometry of [4] into two *commuting* GCS's subject to certain conditions. Since the sigma model we discuss reduces to that of [4] when auxiliary fields are integrated out, we expect to find left and right GCS's that together describe a generalized Kähler geometry. There are many possible sets of auxiliary fields, however, and it is not a priori clear that the ones we choose are the correct "coordinates" for such a description.

For a "toy model" based on semi-chiral multiplets only, we discover that the underlying geometry is indeed a generalized Kähler geometry. This model is not a sigma model proper; nevertheless the second supersymmetry transformations are determined in terms of the GCS's corresponding to the complex structure and to the Kähler form.

For the full model based on semi-chiral and semi-antichiral multiplets we uncover two GCS's corresponding to a second left and a second right supersymmetry. Although the corresponding complex structures do not commute, the GCS's do. Our model thus satisfies part of the requirement for a generalized Kähler geometry, and we subsequently verify that the metric on $T \oplus T^*$ formed as the product of the two GCS's, is idempotent. The underlying GCG is thus the expected generalized Kähler geometry. Further, the metric, the B-field, and the GCS's are all determined by one and the same potential K.

The literature on GCG in physics is not yet very extensive. The geometry has been discussed in connection to generalized Calabi-Yau geometries in [12], a short review of the sigma-model application appeared in [13], applications to topological sigma-models was published in [14, 15] and other sigma model applications were discussed in [16, 17].

The organization of our presentation is as follows: In section 2 we introduce supersymmetric nonlinear sigma-models and recapitulate the features of bihermitean geometry.

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Section 3 contains a brief introduction to generalized complex geometry and section 4 contains an even briefer introduction to sigma model realizations of GCS. In section 5 we introduce the basics of semi-(anti)chiral multiplets and section 6 contains various actions for them. Section 7 shows how the GCS's emerge for the models discussed in section 6. Section 8 eight, finally, contains our conclusions and an outlook.

2. Sigma models

A nonlinear sigma model is a theory of maps

$$X^\mu(\xi) : \mathcal{M} \to \mathcal{T}\,, \tag{2.1}$$

where ξ^i are coordinates on $\mathcal{M}$ and X^μ coordinates on the target space $\mathcal{T}$. Classical solutions are found by extremizing the action.

$$S = \int d\xi\ \partial_i X^\mu\, g_{\mu\nu}(X)\, \partial^i X^\nu\,, \tag{2.2}$$

where the symmetric tensor $g_{\mu\nu}$ is identified with a metric on $\mathcal{T}$. The corresponding geometry is Riemannian for the bosonic model, but becomes complex when we impose enough supersymmetry.

Supersymmetry is introduced by replacing the X^μ's by superfields:

$$X^\mu(\xi) \to \phi^\mu(\xi,\theta)\,. \tag{2.3}$$

We shall be interested in a two dimensional $\mathcal{M}$, the dimension relevant for string theory. There are (at least) two special features in $D = 2$. First, there can be different amounts of supersymmetry in the left and right moving sectors denoted $\mathcal{N} = (p,q)$ supersymmetry. Second, if parity breaking terms are allowed, the background may contain an antisymmetric $B_{\mu\nu}$-field. For $\mathcal{N} = (2,2)$, the supersymmetric action written in terms of real $\mathcal{N} = (1,1)$ superfields reads (with spinorial indices $\alpha = \{+,-\}$)

$$S = \int d^2\xi\, d^2\theta\, D_+\phi^\mu E_{\mu\nu}(\phi) D_-\phi^\nu\,, \tag{2.4}$$

where $E_{\mu\nu}(\phi) \equiv g_{\mu\nu}(\phi) + B_{\mu\nu}(\phi)$. This action has manifest $\mathcal{N} = (1,1)$ supersymmetry without any additional restrictions on the target space geometry. In [4], Gates, Hull and Roček showed that it has additional nonmanifest supersymmetries,

$$\delta\phi^\mu = \varepsilon^+ D_+\phi^\nu \mathbb{J}^{(+)\nu}_{\mu} + \varepsilon^- D_-\phi^\nu \mathbb{J}^{(-)\nu}_{\mu}\,, \tag{2.5}$$

(where $\{(+),(-)\}$ are labels and *not* spinor indices) provided that the following conditions are fulfilled

- Both the J's are *almost complex structures*, i.e. $\mathbb{J}^{(\pm)2} = -1$.
- They are *integrable*, i.e. their Nijenhuis tensors vanish

$$\mathcal{N}^{(\pm)\rho}_{\mu\nu} \equiv \mathbb{J}^{(\pm)\lambda}_{\mu} \partial_{[\lambda} \mathbb{J}^{(\pm)\rho}_{\nu]} - (\mu \leftrightarrow \nu) = 0\,. \tag{2.6}$$

- The metric is *hermitean* with respect to both complex structures, i.e. it is preserved by both structures:

$$\mathbb{J}^{(\pm)t} g \mathbb{J}^{(\pm)} = g\,.$$

- The $\mathbb{J}$'s are *covariantly constant* with respect to a torsionful connection:

$$\nabla^{(\pm)} \mathbb{J}^{(\pm)} = 0\,,$$

  where $\nabla^{(\pm)} \equiv \nabla^0 \pm g^{-1} H$ is the Levi-Civita connection plus(minus) the completely antisymmetric torsion given by the field-strength $H = dB$. (There are alternative, equivalent descriptions; see, e.g., [18].)

The above conditions represent a bihermitean target space geometry with a B-field, and result from requiring invariance of the action (2.4) under the transformations (2.5) as well as closure of the algebra of these transformations. Closure is only achieved on-shell, however. Only under the special condition that the two complex structures commute does the algebra close off-shell. In that case there is a manifestly $\mathcal{N} = (2,2)$ action for the model, given in terms of chiral and twisted chiral $\mathcal{N} = (2,2)$ superfields [4].

If one is willing to introduce additional (auxiliary) spinorial $\mathcal{N} = (1,1)$ superfields, it is known how to accomodate noncommutativity of the complex structures for a special case which also has a manifest $\mathcal{N} = (2,2)$ formulation. This case corresponds to the semi-(anti)chiral superfields [11] that we discuss below.

An interesting question is thus: What is the most general $\mathcal{N} = (2,2)$ sigma model with off-shell closure of the algebra, and what is the corresponding geometry? In asking this we have in mind an extension of the model similar to the semi-chiral models, i.e. to include additional fields to allow off-shell closure in the usual "auxiliary field" pattern and a geometry that includes these fields.

The GCG does contain the bihermitean geometry as a special case and thus seems a promising candidate. We therefore turn to a brief description of the GCG.

3. Generalized complex geometry

To understand the generalization, let us first briefly look at some aspects of the definition of the ordinary complex structure. The features we need are that an almost complex structure J on a d-dimensional manifold $\mathcal{T}$ is a map from the tangent bundle $J : T \to T$ that squares to minus the identity $J^2 = -1$. With these properties $\pi_\pm \equiv \frac{1}{2}(1 \pm iJ)$ are projection operators, and we may ask when they define integrable distributions. The condition for this is that

$$\pi_\mp[\pi_\pm X, \pi_\pm Y] = 0 \tag{3.1}$$

for $X, Y \in T$ and $[,]$ the usual Lie-bracket on T. This relation is equivalent to the vanishing of the Nijenhuis tensor $\mathcal{N}(J)$, as defined in (2.6).

To define GCG, we turn our attention from the tangent bundle $T(\mathcal{T})$ to the sum of the tangent bundle and the co-tangent bundle $T \oplus T^*$. (Note that the structure group

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of this bundle is $SO(d,d)$, the string theory T-duality group[1]). We write an element of $T \oplus T^*$ as $X + \xi$ with the vector $X \in T$ and the one-form $\xi \in T^*$. The natural pairing $(X+\xi, X+\xi) = \imath_X \xi$ gives a metric $\mathcal{I}$ on $T \oplus T^*$ as $X + \xi$, which in a coordinate basis (∂_μ, dx^ν) reads

$$\mathcal{I} = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}. \tag{3.2}$$

In the definition of a complex structure above we made use of the Lie-bracket on T. To define a generalised complex structure we will need a bracket on $T \oplus T^*$. The relevant bracket is the the skew-symmetric *Courant bracket* [20] defined by[2]

$$[X+\xi, Y+\eta]_c \equiv [X,Y] + \pounds_X \eta - \pounds_Y \xi - \tfrac{1}{2} d(\imath_X \eta - \imath_Y \xi)\,. \tag{3.3}$$

This bracket equals the Lie-bracket on T and vanishes on T^*. The most important property for us in the context of sigma-models is that its group of automorphisms is not only $Diff(\mathcal{T})$ but also *b-transforms* defined by closed two-forms b,

$$e^b(X+\xi) \equiv X + \xi + \imath_X b\,, \tag{3.4}$$

namely,

$$[e^b(X+\xi), e^b(Y+\eta)]_c = e^b[X+\xi, Y+\eta]_c\,. \tag{3.5}$$

A *generalized almost complex structure* is an endomorphism $\mathcal{J} : T \oplus T^* \to T \oplus T^*$ that satisfies $\mathcal{J}^2 = -1_{2d}$ and preserves the natural metric $\mathcal{I}$, $\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}$. The projection operators $\Pi_\pm \equiv \frac{1}{2}(1 \pm i\mathcal{J})$ are then used to define integrability (making $\mathcal{J}$ a generalized complex structure) as

$$\Pi_\mp [\Pi_\pm (X+\xi), \Pi_\pm (Y+\eta)]_c = 0\,. \tag{3.6}$$

In a coordinate basis $\mathcal{J}$ is representable as

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & K \end{pmatrix}, \tag{3.7}$$

where $J : T \to T$, $P : T^* \to T$, $L : T \to T^*$, $K : T^* \to T^*$. The condition $\mathcal{J}^2 = -1_{2d}$ will impose the conditions

$$\begin{aligned} J^2 + PL &= -1_d \\ JP + PK &= 0 \\ KL + LJ &= 0 \end{aligned}$$

[1] This connection between $T \oplus T^*$ was early on made in [19]

[2] It does not in general satisfy the Jacobi identity; had it satisfied the Jacobi identity $(T \oplus T^*, [,]_c)$ would have formed a Lie algebroid. It *does* satisfy the Jacobi identity on subbundles $L \subset T \oplus T^*$ that are Courant involutive and isotropic with respect to $\mathcal{I}$ that is, subbundles that close under the bracket and whose sections are null with respect the metric $\mathcal{I}$ (for details see chapter 3 in [8]), but fails to do so in general. It fails in an interesting way which leads to the definition of a Courant algebroid [8].

Another physical context where the Courant bracket naturally arises is that of anomaly-freedom of generalized currents recently discussed in [21].

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$$LP + K^2 = -1_d \,, \tag{3.8}$$

hermiticity of $\mathcal{I}$ implies

$$\begin{aligned} K &= -J^t \\ P^t &= -P \\ L^t &= -L \end{aligned} \tag{3.9}$$

and (3.6) will impose differential conditions on J, P, L and K (For their explicit form, see [10]).

An ordinary complex structure J corresponds to the GCS

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}, \tag{3.10}$$

and a symplectic structure ω corresponds to[3]

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{3.11}$$

A b-transform acts as follows

$$\mathcal{J}_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mathcal{J} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}. \tag{3.12}$$

The general situation is illustrated in figure 1.

A useful property for calculations is that locally (in an open set around a regular point) a manifold which admits a generalized complex structure may be brought to look like an open set in $\mathbb{C}^k$ times an open set in (R^{2d-2k}, ω), where ω is in Darboux coordinates and $\mathbb{C}^k$ in complex (holomorphic and antiholomorphic) coordinates (using diffeomorphisms and b-transforms)[4].

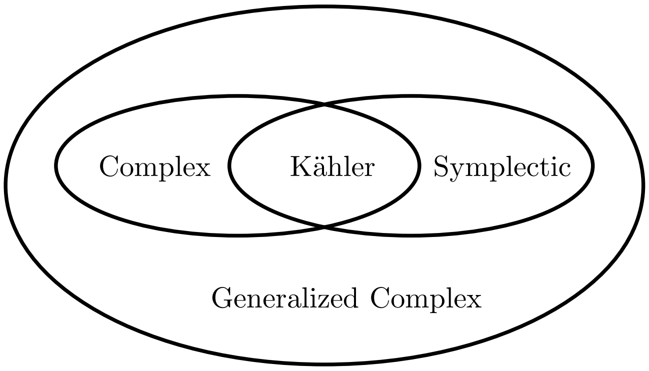

Figure 1: The relation between the different geometries discussed.

The generalized complex geometry is said to be *generalized Kähler* [8] if there exist[5] two commuting generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ such that $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ is a positive

[3]For a generalized complex structure to exist T has to be even-dimensional.

[4]The proof of this, generalizing the Newlander-Nirenberg and the Darboux theorems, may be found in Gualtieri's thesis [8], section 4.7.

[5]See chapter 6 of [8] for a full discussion of generalized Kähler geometry and the relation to bihermitean geometry.

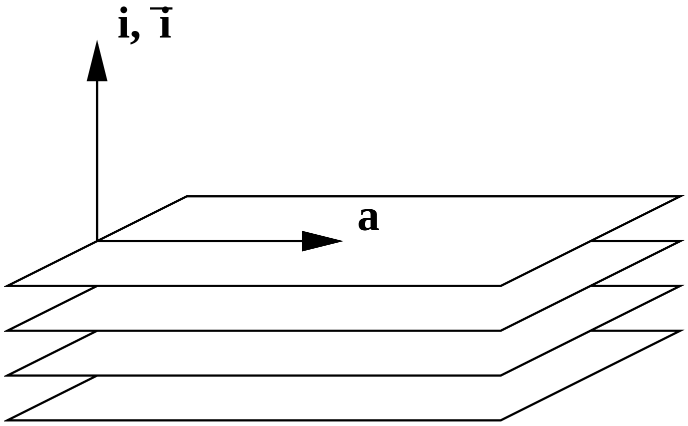

Figure 2: A naive picture of the local foliation in complex coordinates $z^i, \bar{z}^i$ and Darboux coordinates x^a.

definite metric on $T \oplus T^*$. For a Kähler manifold (J, g, ω), using (3.10) and (3.11) one finds the metric

$$\mathcal{G} = -\mathcal{J}_J \mathcal{J}_\omega = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} . \tag{3.13}$$

One of the main results in [8] is that the bihermitean geometry with a B-field discovered in [4] is equivalent to generalized Kähler geometry.

Finally, it is worth mentioning that it is possible to twist the above structure by a closed three-form.

We now turn to the question of how this geometry may be realized in sigma-models.

4. Sigma model realization

In this section we recapitulate the basic set up used to discuss the relation between supersymmetric sigma-models and GCS's in [10, 9].

To realize the GCS in a sigma model, we need a formulation with additional fields $S \in T^*$. We thus consider the following first order action

$$S = \int d^2\xi d^2\theta \left(S_{\mu+} E^{\mu\nu}(\phi) S_{\nu-} - S_{\mu(+} D_{-)}\phi^\mu + D_+\phi^\mu (B-b)_{\mu\nu} D_-\phi^\nu \right) , \tag{4.1}$$

where $E_{\mu\nu} \equiv g_{\mu\nu} + b_{\mu\nu}$ and its inverse may be thought of as open string data:

$$E^{(\mu\nu)} = G^{\mu\nu} , \qquad E^{[\mu\nu]} = \theta^{\mu\nu} . \tag{4.2}$$

In (4.1), $S_{\mu\pm}$ acts as an auxiliary field that extends the model to a sigma model on $T \oplus T^*$, and b is a globally defined two-form that makes it possible to display the b-transform. (Note that the original model (4.1) depends only on $H = dB$, and B is thus typically only locally defined). Eliminating $S_{\mu\pm}$ we recover the action in (2.4). The b-transform is the statement that if in two actions of the form (4.1), $E_{\mu\nu}$ and $\tilde{E}_{\mu\nu}$ differ by a closed two-form $\tilde{b}$, the two actions are equivalent.

The action (4.1) has many interesting limits. For example, if the metric is set to zero, the action describes a supersymmetric version of a Poisson sigma model [22]. In what follows we are not interested in the difference between the B and b-fields, but set them equal each other, so the $\mathcal{N}=(1,1)$ action we study is

$$S=\int d^2\xi d^2\theta \left(S_{\mu+}E^{\mu\nu}(\phi)S_{\nu-}-S_{\mu(+}D_{-)}\phi^{\mu}\right), \tag{4.3}$$

The form of ansatz for the second supersymmetry ($\delta=\delta^{(+)}+\delta^{(-)}$) is determined by a dimensional analysis to be [10]

$$\begin{aligned}\delta^{(\pm)}\phi^{\mu} &= \varepsilon^{\pm}\Lambda^{A}_{\pm}A^{(\pm)\mu}_{A}\\ \delta^{(\pm)}S_{\mu\pm} &= \varepsilon^{\pm}\left(D_{\pm}\Lambda^{A}_{\pm}B^{(\pm)}_{\mu A}+\Lambda^{A}_{\pm}\Lambda^{B}_{\pm}C^{(\pm)}_{\mu AB}\right)\\ \delta^{(\pm)}S_{\mu\mp} &= \varepsilon^{\pm}\left(D_{\pm}\Lambda^{A}_{\mp}M^{(\pm)}_{\mu A}+D_{\mp}\Lambda^{A}_{\pm}N^{(\pm)}_{\mu A}+\Lambda^{A}_{\pm}\Lambda^{B}_{\mp}X^{(\pm)}_{\mu AB}\right),\end{aligned} \tag{4.4}$$

where $\Lambda^{A}_{\pm}\equiv(D_{\pm}\phi^{\mu},\, S_{\mu\pm})$ lie in $T\oplus T^{*}$, and all the coefficients $A^{(\pm)\mu}_{A}, B^{(\pm)}_{\mu A}, C^{(\pm)}_{\mu AB}, M^{(\pm)}_{\mu A}$, $N^{(\pm)}_{\mu A}$, and $X^{(\pm)}_{\mu AB}$ are functions of ϕ. The conditions that follow from invariance of the action and closure of the algebra are of two kinds: algebraic and differential. The two-index coefficients (C and X) typically turn out to be given as derivatives of the one-index coefficients (A, B, M and N), just as the generalized complex structures are given in terms of J, P, L and K, which subsequently obey differential conditions via the integrability requirement. In [10] it is shown that the analogous index coefficients for certain $\mathcal{N}=(2,0)$ sigma-models have a direct interpretation as submatrices of a GCS.

5. Semi-Chiral multiplets

Semi-(anti)chiral superfields are left or right chiral $\mathcal{N}=(2,2)$) multiplets and were introduced in [11]. They are represented by $\mathcal{N}=(2,2)$ superfields that obey "half" the usual chirality constraints. A complex left chiral superfield $\mathbb{X}$ obeys

$$\bar{\mathbb{D}}_{+}\mathbb{X}=0\,, \tag{5.1}$$

and a right antichiral superfield $\mathbb{Y}$ obeys

$$\mathbb{D}_{-}\mathbb{Y}=0\,. \tag{5.2}$$

Decomposing the $\mathcal{N}=(2,2)$ covariant derivatives $(\mathbb{D}_{\pm},\bar{\mathbb{D}}_{\pm})$ into $\mathcal{N}=(1,1)$ derivatives $D_{\pm}$ and extra supercharges $Q_{\pm}$,

$$\begin{aligned}D_{\pm} &= \mathbb{D}_{\pm}+\bar{\mathbb{D}}_{\pm}\,,\\ Q_{\pm} &= i(\mathbb{D}_{\pm}-\bar{\mathbb{D}}_{\pm})\,,\end{aligned} \tag{5.3}$$

and the $\mathcal{N}=(2,2)$ superfields into $\mathcal{N}=(1,1)$ superfields,

$$\varphi\equiv\mathbb{X}| \qquad \Psi_{-}\equiv Q_{-}\mathbb{X}|$$

$$\chi \equiv \mathbb{Y}| \qquad \Upsilon_+ \equiv Q_+\mathbb{Y}|\,, \tag{5.4}$$

the conditions (5.1), (5.2) become

$$\begin{array}{llll} Q_+\varphi = iD_+\varphi\,, & Q_-\varphi = \Psi_-\,, & Q_+\Psi_- = iD_+\Psi_-\,, & Q_-\Psi_- = -i\partial_{=}\varphi\,, \\ Q_+\bar\varphi = -iD_+\bar\varphi\,, & Q_-\bar\varphi = \bar\Psi_-\,, & Q_+\bar\Psi_- = -iD_+\bar\Psi_-\,, & Q_-\bar\Psi_- = -i\partial_{=}\bar\varphi\,, \\ Q_-\chi = -iD_-\chi\,, & Q_+\chi = \Upsilon_+\,, & Q_-\Upsilon_+ = -iD_-\Upsilon_+\,, & Q_+\Upsilon_+ = -i\partial_{+\!\!+}\chi\,, \\ Q_-\bar\chi = iD_-\bar\chi\,, & Q_+\bar\chi = \bar\Upsilon_+\,, & Q_-\bar\Upsilon_+ = iD_-\bar\Upsilon_+\,, & Q_+\bar\Upsilon_+ = -i\partial_{+\!\!+}\bar\chi\,, \end{array} \tag{5.5}$$

where vector indices are denoted by pairs of spinor indices: $\{+\!\!+, =\}$.

For p left-chiral superfields $\mathbb{X}^a$, $a = 1, \ldots, p$ and p' right antichiral superfields $\mathbb{Y}^{a'}$, $a' = 1, \ldots, p'$, it is convenient to introduce the notation[6] $A = a, \bar a$ and $A' = a', \bar a'$. In this notation, we define the complex structures on the subspaces

$$J^A_{\ B} \equiv \begin{pmatrix} i\delta^b_a & 0 \\ 0 & -i\delta^{\bar b}_{\bar a} \end{pmatrix} \qquad J^{A'}_{\ B'} \equiv \begin{pmatrix} i\delta^{b'}_{a'} & 0 \\ 0 & -i\delta^{\bar b'}_{\bar a'} \end{pmatrix}, \tag{5.6}$$

the relations (5.5) can be written as

$$\begin{array}{llll} Q_+\varphi^A = J^A_{\ B} D_+\varphi^B\,, & Q_+\Psi^A_- = J^A_{\ B} D_+\Psi^B_-\,, & Q_+\chi^{A'} = \Upsilon^{A'}_+\,, & Q_+\Upsilon^{A'}_+ = -i\partial_{+\!\!+}\chi^{A'}\,, \\ Q_-\chi^{A'} = -J^{A'}_{\ B'} D_-\chi^{B'}\,, & Q_-\Upsilon^{A'}_+ = -J^{A'}_{\ B'} D_-\Upsilon^{B'}\,, & Q_-\varphi^A = \Psi^A_-\,, & Q_-\Psi^A_- = -i\partial_{=}\varphi^A\,. \end{array} \tag{5.7}$$

Despite the similarities with the geometrical discussion in section 3 we cannot identify the generalized Kähler geometry of our model directly from (5.7). If we restrict ourselves to Q_+ transformations, then the A-directions correspond to a transverse complex structure and the A'-directions to symplectic directions, thus indicating the relation to the generalized complex geometry. The same can be done for the Q_- transformations. However, an off-shell formulation never contains the full information about the geometry (e.g., the metric), and thus an analysis of possible actions and their invariances is needed.

6. Actions

6.1 A topological model

A general action that depends only on $\mathbb{X}$ and $\bar{\mathbb{X}}$ gives a topological example:

$$S_{\mathbb{X}} = \int d^2\xi d^2\theta d^2\bar\theta K(\mathbb{X}^a, \bar{\mathbb{X}}^{\bar a}) = \int d^2\xi \mathbb{D}^2\bar{\mathbb{D}}^2 K(\mathbb{X}^a, \bar{\mathbb{X}}^{\bar a})\,. \tag{6.1}$$

Reducing to $\mathcal{N} = (1,1)$ using the results in section 5 in the standard fashion (see e.g. [4]), we find

$$S_{\mathbb{X}} = -\frac{i}{4}\int d^2\xi D^2(\omega_{AB} D_+\varphi^A \Psi^B_-) = -\frac{i}{4}\int d^2\xi D^2(D_+\varphi^A S_{A-})\,, \tag{6.2}$$

where we have redefined [11]

$$S_{A-} = \omega_{AB}\Psi^B_-\,, \tag{6.3}$$

[6] Not to be confused with the previous usage of the index A in section 4.

and introduced the symplectic form

$$\omega_{AB} \equiv \begin{pmatrix} 0 & iK_{a\bar{b}} \\ -iK_{\bar{a}b} & 0 \end{pmatrix} . \tag{6.4}$$

6.2 Sigma models

To construct a sigma model, we need to introduce an equal number of semi-chiral and semi-antichiral fields. This ensures that all the auxiliary superfields can be eliminated. The most general manifest $\mathcal{N} = (2,2)$ action with p left chiral multiplets $\mathbb{X}^a$ and $p = p'$ right chiral multiplets $\mathbb{Y}^{a'}$ is

$$\int d^2\xi d^2\theta d^2\bar{\theta} K(\mathbb{X}^A, \mathbb{Y}^{A'}) = \int d^2\xi \mathbb{D}^2 \bar{\mathbb{D}}^2 K(\mathbb{X}^A, \mathbb{Y}^{A'}) = -\frac{i}{4} \int d^2\xi D^2 Q_+ Q_- K(\mathbb{X}^A, \mathbb{Y}^{A'}) . \tag{6.5}$$

We reduce to $\mathcal{N} = (1,1)$ by acting with $Q_\pm$ and keeping the part independent of the second θ. The result is

$$\begin{aligned} S = \frac{1}{4} \int d^2\xi \; D^2 \Big(& -iD_+\varphi^A m_{AA'} D_- \chi^{A'} - i\Upsilon_+^{A'} n_{A'A} \Psi_-^A + \Psi_-^A (-2i\omega_{AB} D_+\varphi^B + p_{AB'} D_+ \chi^{B'}) - \\ & -\Upsilon_+^{A'} (q_{A'B} D_- \varphi^B + 2i\omega_{A'B'} D_- \chi^{B'}) \Big) , \end{aligned} \tag{6.6}$$

where the $2p \times 2p$ matrices are

$$\begin{aligned} m_{AA'} &= J^B_A K_{B'B} J^{B'}_{A'} = \begin{pmatrix} K_{aa'} & -K_{a\bar{a}'} \\ -K_{\bar{a}a'} & K_{\bar{a}\bar{a}'} \end{pmatrix} \\ n_{A'A} &= K_{A'A} = \begin{pmatrix} K_{a'a} & K_{a'\bar{a}} \\ K_{\bar{a}'a} & K_{\bar{a}'\bar{a}} \end{pmatrix} \\ -2i\omega_{AB} &= -iJ^C_{[A} K_{B]C} = \begin{pmatrix} 0 & 2K_{a\bar{b}} \\ -2K_{a\bar{b}} & 0 \end{pmatrix} \\ -2i\omega_{A'B'} &= -iJ^{C'}_{[A'} K_{B']C'} = \begin{pmatrix} 0 & 2K_{a'\bar{b}'} \\ -2K_{a'\bar{b}'} & 0 \end{pmatrix} \\ p_{AA'} &= -iJ^C_A K_{CA'} = \begin{pmatrix} K_{aa'} & K_{a\bar{a}'} \\ -K_{\bar{a}a'} & -K_{\bar{a}\bar{a}'} \end{pmatrix} \\ q_{A'A} &= iJ^{C'}_{A'} K_{C'A} = \begin{pmatrix} -K_{a'a} & -K_{a'\bar{a}} \\ K_{\bar{a}'a} & K_{\bar{a}'\bar{a}} \end{pmatrix} . \end{aligned} \tag{6.7}$$

Here ω_{AB} and $\omega_{A'B'}$ are symplectic structures.

Since (6.6) is not directly comparable to the action (4.1) that was previously used to discuss relations to GCG, we now show how to relate them. First we rewrite (6.6) as[7]

$$S = \frac{1}{4} \int d^2\xi D^2 \left(\mathbf{\Psi}_+^{\mathbf{t}} \mathbb{N} \mathbf{\Psi}_- + \mathbf{\Psi}_+^{\mathbf{t}} \mathbb{Q} D_- \mathbf{\Phi} + \mathbf{\Psi}_-^{\mathbf{t}} \mathbb{P} D_+ \mathbf{\Phi} + D_+ \mathbf{\Phi}^{\mathbf{t}} \mathbb{M} D_- \mathbf{\Phi} \right) , \tag{6.8}$$

[7] Here t denotes transpose.

where we have defined the $4p$ dimensional column vectors

$$\mathbf{\Psi}_{\pm} \equiv \begin{pmatrix} \Psi^{A}_{\pm} \\ \Upsilon^{A'}_{\pm} \end{pmatrix}, \qquad \mathbf{\Phi} \equiv \begin{pmatrix} \varphi^{A} \\ \chi^{A'} \end{pmatrix}, \tag{6.9}$$

and the $4p \times 4p$ matrices

$$\mathbb{N} \equiv -i \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}, \qquad \mathbb{Q} \equiv - \begin{pmatrix} 0 & 0 \\ q & 2i\omega' \end{pmatrix},$$

$$\mathbb{M} \equiv -i \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, \qquad \mathbb{P} \equiv \begin{pmatrix} -2i\omega & p \\ 0 & 0 \end{pmatrix}. \tag{6.10}$$

In the definition (6.9), we have introduced two new sets of fields Ψ^{a}_{+} and $\Upsilon^{a'}_{-}$. From the form of the matrices in (6.10) it follows that they do not appear in the action (6.8), but we shall need them below in discussing the relation to the action (4.3).

To compare (6.8) to the action (4.3) we redefine the auxiliary fields $\mathbf{\Psi}_{\pm}$ in two steps. First we shift

$$\mathbf{\Psi}_{\pm} = \hat{\mathbf{\Psi}}_{\pm} + \mathbb{A}^{(\pm)} D_{\pm} \mathbf{\Phi}\,. \tag{6.11}$$

This changes the Lagrangian in (6.8) into

$$\begin{aligned} L \,=\, & \hat{\mathbf{\Psi}}^{\mathbf{t}}_{+} \mathbb{N} \hat{\mathbf{\Psi}}_{-} + \hat{\mathbf{\Psi}}^{\mathbf{t}}_{+} (\mathbb{Q} + \mathbb{N}\mathbb{A}^{(-)}) D_{-}\mathbf{\Phi} + \hat{\mathbf{\Psi}}^{\mathbf{t}}_{-} (\mathbb{P} - \mathbb{N}^{t}\mathbb{A}^{(+)}) D_{+}\mathbf{\Phi} \\ & + D_{+}\mathbf{\Phi}^{\mathbf{t}} \left(\mathbb{M} + \mathbb{A}^{(+)t}\mathbb{N}\mathbb{A}^{(-)} - \mathbb{P}^{t}\mathbb{A}^{(-)} + \mathbb{A}^{(+)t}\mathbb{Q} \right) D_{-}\mathbf{\Phi}\,. \end{aligned} \tag{6.12}$$

Next, to compare to (4.3), we want to choose the matices $\mathbb{A}^{(\pm)}$ to remove the last term in (6.12). The form of the matrices (6.10) makes this impossible, however. This can be traced back to the properties of the left and right chiral multiplets we use; these contain only the plus or the minus auxiliary $\mathcal{N} = (1,1)$ spinor superfields. Thus the number of auxiliary fields does not match the number of coordinate fields. There are two ways to proceed: Either we compare to the form of the action that results from (4.3) after integrating out some of the S-fields or we "reinstate" the missing auxiliaries at the $\mathcal{N} = (1,1)$ level by adding a trivial Lagrangian of the type $\mathcal{L}_{Extra} = \Psi^{a}_{+}\delta_{aa'}\Upsilon^{a'}_{-}$ in (6.8). It is for this purpose the fields Ψ^{a}_{+} and $\Upsilon^{a'}_{-}$ were included in (6.9). Their supersymmetry transformations are simply

$$\begin{aligned} Q_{+}\Psi^{a}_{+} &= iD_{+}\Psi^{a}_{+}\,,\ Q_{+}\Upsilon^{a'}_{-} = iD_{+}\Upsilon^{a'}_{-}\,, \\ Q_{-}\Psi^{a}_{+} &= iD_{-}\Psi^{a}_{+}\,,\ Q_{-}\Upsilon^{a'}_{-} = iD_{-}\Upsilon^{a'}_{-}\,, \end{aligned} \tag{6.13}$$

and their algebra closes off-shell. We first discuss this case.

The effect of $\mathcal{L}_{Extra}$ is to change $\mathbb{N} \to \widetilde{\mathbb{N}}$ in (6.12), where

$$\widetilde{\mathbb{N}} \equiv -i \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix} \tag{6.14}$$

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is now invertible. We may thus choose $\mathbb{A}^{(\pm)}$ such that $\mathbb{M}+\mathbb{A}^{(+)t}\widetilde{\mathbb{N}}\mathbb{A}^{(-)}-\mathbb{P}^t\mathbb{A}^{(-)}+\mathbb{A}^{(+)t}\mathbb{Q}=0$. Next we transform

$$\begin{aligned}\widetilde{\mathbf{\Psi}}_+ &= \left(\mathbb{Q}+\widetilde{\mathbb{N}}\mathbb{A}^{(-)}\right)^t\hat{\mathbf{\Psi}}_+\\ \widetilde{\mathbf{\Psi}}_- &= \left(\mathbb{P}-\widetilde{\mathbb{N}}^t\mathbb{A}^{(+)}\right)^t\hat{\mathbf{\Psi}}_-\,. \end{aligned} \tag{6.15}$$

With these transformations, the action (6.8)is brought to the form

$$S=\int d^2\xi d^2\theta\left(\widetilde{\mathbf{\Psi}}^{\mathbf{t}}_+\mathbb{E}^{-1}\widetilde{\mathbf{\Psi}}_- + \widetilde{\mathbf{\Psi}}^{\mathbf{t}}_{(+}D_{-)}\mathbf{\Phi}\right), \tag{6.16}$$

with

$$\mathbb{E}^{-1}\equiv\left(\mathbb{Q}+\widetilde{\mathbb{N}}\mathbb{A}^{(-)}\right)^{-1}\widetilde{\mathbb{N}}\left[\left(\mathbb{P}-\widetilde{\mathbb{N}}^t\mathbb{A}^{(+)}\right)^t\right]^{-1}. \tag{6.17}$$

This is the action (4.3) in complex coordinates.

From (6.17), we may thus read off the expression for $E^{\mu\nu}$, whose inverse gives us the background metric and antisymmetric field. Clearly both of these are given in terms of derivatives of K. The transformations (4.4) are then found by applying (5.5) and (6.13) to the fields $\mathbf{\Phi}$ and $\widetilde{\mathbf{\Psi}}_\pm$.

In the alternative approach where we don't add $\mathcal{L}_{Extra}$, we choose $\mathbb{A}^{(\pm)}$ to have some convenient form and then make a redefinition as in (6.15). Independent of our choice, there will always be a nonzero $D_+\mathbf{\Phi}D_-\mathbf{\Phi}$-term. To compare to (4.3) we must integrate out some $S_\pm$ to generate such a term.

Finally, integrating out $\widetilde{\mathbf{\Psi}}_\pm$ from (6.16) or $\mathbf{\Psi}_\pm$ from (6.8), we find the sum of the metric and antisymmetric B-field to be given in terms of the potential K as

$$\mathbb{E}=\mathbb{M}+\mathbb{P}^t\widetilde{\mathbb{N}}^{-1}\mathbb{Q}\,. \tag{6.18}$$

7. Geometric interpretation

In this section we show that the transformations (5.5) and (6.13) have a simple interpretation in terms of the GCG described in section 3. As mentioned there, we expect to find a generalized Kähler geometry.

7.1 A topological model

We first discuss the topological model (6.2). Although this is not a proper sigma model, we expect it to be related to GCG in a manner analogous to the toy-model discussed in [10]. The second supersymmetry transformations are, from (5.7) and (6.3),

$$\begin{aligned}\delta^{(+)}\varphi^A &= \varepsilon^+D_+\varphi^BJ^A_{\ B}\,,\\ \delta^{(-)}\varphi^A &= -\varepsilon^-\omega^{AB}S_{B-}\,,\\ \delta^{(+)}S_{A-} &= -\varepsilon^+D_+S_{B-}J^B_{\ A}+\omega_{AB}(\varepsilon^+J^B_{\ C}D_+\varphi^E-\delta^B_C\delta^{(+)}\varphi^E)\partial_E(\omega^{CD})S_{D-}\,,\\ \delta^{(-)}S_{A-} &= -i\varepsilon^-\omega_{AB}\partial_{=}\varphi^B+\omega^{BD}\partial_C(\omega_{AB})\delta^{(-)}\varphi^CS_{D-}\,, \end{aligned} \tag{7.1}$$

where ω^{AB} is the inverse matrix of ω_{AB}. Closure of the algebra and invariance of the action are not issues here; they are satisfied by construction (and are easily checked). These transformations represent a special case of the transformations (4.4) (in complex coordinates). As discussed in [10], the lowest tensor terms in the transformations typically give the generalized complex structure $\mathcal{J}$ while the higher terms are derivatives of lower ones needed for closure of the algebra, and geometrically ensure that closure of the algebra (and invariance of the action) leads to integrability of $\mathcal{J}$. Here we already know that the GCS's will be integrable, since we start from a manifest $\mathcal{N} = (2, 2)$ theory. We thus focus on identifying the GCS's and verify that they represent a generalized Kähler geometry. We find that also here the lowest tensor terms suffice to identify the GCS. We then verify that it has the expected properties described in [8]. From the terms in (7.1) not containing derivatives of the metric we identify the GCS

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}, \tag{7.2}$$

(with J as defined in (5.6)) corresponding to the $(+)$ transformations. Since $\mathcal{J}_J$ is constant, integrability is immediate. The $(-)$-transformations instead define

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \tag{7.3}$$

where ω_{AB} is the the Kähler form corresponding to the Kähler metric $g = \omega J$, which in turn defines the metric $\mathcal{G}$ on $T \oplus T^*$;

$$\mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}, \tag{7.4}$$

We have thus uncovered the simplest case of generalized Kähler geometry[8] as defined in [8], and as a final check we verify (3.13):

$$-\mathcal{J}_J \mathcal{J}_\omega = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} = \mathcal{G}\,. \tag{7.5}$$

We now turn to the full model in the form of the action (6.6).

7.2 Sigma models

Using the field-redefinitions

$$\begin{aligned} u^{AA'} S_{A'-} &\equiv \Psi_-^A - i\, u^{AA'}(q_{A'B} D_- \varphi^B + 2i\omega_{A'B'} D_- \chi^{B'}) \\ u^{AA'} S_{A+} &\equiv \Upsilon_+^{A'} - i\, u^{AA'}(-2i\omega_{AB} D_+ \varphi^B + p_{AB'} D_+ \chi^{B'})\,, \end{aligned} \tag{7.6}$$

where $u^{AA'}$ is the inverse of $n_{A'A}$, the action (6.6) reads

[8]It represents the ordinary Kähler geometry (J, g, ω).

$$
\begin{aligned}
S = -\frac{i}{4}\int d^2\xi\, D^2 \Big(& D_+\varphi^A(m_{AA'} + 4\omega_{BA}u^{BB'}\omega_{B'A'})D_-\chi^{A'} - \\
& -2iD_+\varphi^A\omega_{BA}u^{BB'}q_{B'C}D_-\varphi^C + D_+\chi^{A'}p_{AA'}u^{AB'}q_{B'B}D_-\varphi^B + \\
& +2iD_+\chi^{A'}p_{AA'}u^{AB'}\omega_{B'C'}D_-\chi^{C'} + S_{A+}u^{AA'}S_{A'-}\Big) \\
\equiv -\frac{i}{4}\int d^2\xi D^2 \Big(& D_+\mathbf{\Phi}^t\mathbb{E}D_-\mathbf{\Phi} + S_{A+}u^{AA'}S_{A'-}\Big)\,,
\end{aligned}
\tag{7.7}
$$

with $\mathbb{E}$ the sum of the metric and the B-field,

$$
\mathbb{E} = \begin{pmatrix} 2i\omega uq & m - 4\omega u\omega' \\ p^t uq & 2ip^t u\omega' \end{pmatrix}.
\tag{7.8}
$$

This is an explicit form of the relation (6.18) above.

The second (+) supersymmetry transformations in (5.5) read, with dots representing φ and χ-derivative terms,

$$
\begin{aligned}
\delta^{(+)}\varphi^A &= \varepsilon^+ D_+\varphi^B J^A_B \\
\delta^{(+)}\chi^{A'} &= \varepsilon^+ u^{AA'}\Big(S_{A+} + 2\omega_{AB}D_+\varphi^B + ip_{AB'}D_+\chi^{B'}\Big) \\
\delta^{(+)}S_{A+} &= \varepsilon^+\Big(ip_{AA'}u^{BA'}D_+S_{B+} + (n_{A'A} - p_{AB'}u^{BB'}p_{BA'})D_+^2\chi^{A'} + \\
&\qquad + 2(\omega_{AB}J^B_C + p_{AA'}u^{BA'}\omega_{BC})D_+^2\varphi^C\Big) + \cdots \\
\delta^{(+)}S_{A'-} &= \varepsilon^+\Big(-i(n_{A'A}J^A_B u^{BB'}q_{B'C} - q_{A'A}J^A_C - 4i\omega_{A'B'}u^{BB'}\omega_{BC})D_-D_+\varphi^B - \\
&\qquad - 2(n_{A'A}J^A_B u^{BB'}\omega_{B'C'} + i\omega_{A'B'}u^{BB'}p_{BC'})D_-D_+\chi^{C'} - \\
&\qquad - 2\omega_{A'B'}u^{BB'}D_-S_{B+} + n_{A'A}J^A_B u^{BB'}D_+S_{B'-}\Big) + \cdots\,.
\end{aligned}
\tag{7.9}
$$

From this we read off the following $8p \times 8p$ GCS

$$
\mathcal{J}^{(+)} = \begin{pmatrix}
J & 0 & 0 & 0 \\
2u^t\omega & iu^t p & u^t & 0 \\
-2(\omega J + ipu^t\omega) & -(n - pu^t p) & -ipu^t & 0 \\
i(-nJu^t q + qJ + 4i\omega' u^t\omega) & 2(nJu\omega' - i\omega' u^t p) & -2\omega' u^t & nJu
\end{pmatrix}.
\tag{7.10}
$$

It may be verified that $(\mathcal{J}^{(+)})^2 = -\mathbf{1}$, independent of the actual form (6.7) of the submatrices[9]. Finally, the form of the action (7.7) shows that on S-shell, i.e. eliminating $S_\pm$, we recover the usual second order action studied by Gates, Hull and Roček in [4]. We must thus also recover the corresponding second supersymmetry transformations

$$
\delta^{(\pm)}\mathbf{\Phi} = \mathbb{J}^{(\pm)}\varepsilon^\pm D_\pm\mathbf{\Phi}\,,
\tag{7.11}
$$

where $\mathbf{\Phi}$ is defined in (6.9). Setting $S_\pm = 0$ in (7.9) and comparing we see that the upper left hand submatrix $\mathbb{J}^{(+)}$ in (7.10) must be an (ordinary) complex structure. It is indeed straightforward to show, using (6.7), that

$$
(\mathbb{J}^{(+)})^2 = \begin{pmatrix} J & 0 \\ 2u^t\omega & iu^t p \end{pmatrix}^2 = -\begin{pmatrix}
\delta^a_b & 0 & 0 & 0 \\
0 & \delta^{\bar a}_{\bar b} & 0 & 0 \\
0 & 0 & \delta^{a'}_{b'} & 0 \\
0 & 0 & 0 & \delta^{\bar a'}_{\bar b'}
\end{pmatrix}.
\tag{7.12}
$$

[9]Although the entries in the matrix (7.10) may be considerably simplified using (6.7).

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Here it is important to note that this part of the GCS (7.10) represents the full φ and χ transformations, i.e. no derivative corrections were left out in (7.9) in the transformations of these fields.

We now repeat the preceeding analysis for the second $(-)$ supersymmetry. The transformations in (5.5) read

$$\begin{aligned}
\delta^{(-)}\varphi^A &= \varepsilon^- u^{AA'}\Big(S_{A'-} + iq_{A'B}D_-\varphi^B - 2\omega_{A'B'}D_-\chi^{B'}\Big)\\
\delta^{(-)}\chi^{A'} &= -\varepsilon^- D_-\chi^{B'} J^{A'}_{\ B'}\\
\delta^{(-)}S_{A+} &= \varepsilon^-\Big(2(n_{A'A}J^{A'}_{\ B'}u^{BB'}\omega_{BC} + i\omega_{AB}u^{BB'}q_{B'C})D_+D_-\varphi^C +\\
&\quad + (in_{A'A}J^{A'}_{\ B'}u^{BB'}p_{BC'} - 4\omega_{AB}u^{BB'}\omega_{B'C'} - ip_{AB'}J^{B'}_{\ C'})D_+D_-\chi^{C'} +\\
&\quad + 2\omega_{AB}u^{BB'}D_+S_{B'-} - n_{A'A}J^{A'}_{\ B'}u^{BB'}D_-S_{B+}\Big) + \cdots\\
\delta^{(-)}S_{A'-} &= \varepsilon^-\Big((-n_{A'C} + q_{A'A}u^{AB'}q_{B'C})D_-^2\varphi^C + iq_{A'A}u^{AB'}D_-S_{B'-} +\\
&\quad + 2(iq_{A'A}u^{AB'}\omega_{B'C'} - \omega_{A'B'}J^{B'}_{\ C'})D_-^2\chi^{C'}\Big) + \cdots .
\end{aligned} \tag{7.13}$$

From this we read off the following GCS

$$\mathcal{J}^{(-)} = \begin{pmatrix} iuq & -2u\omega' & 0 & -u \\ 0 & -J' & 0 & 0 \\ -2(n^tJ'u^t\omega + i\omega uq) & -i(n^tJ'u^tp + 4i\omega u\omega' - pJ') & -n^tJ'u^t & 2\omega u \\ n - quq & -2(iqu\omega' - \omega'J') & 0 & -iqu \end{pmatrix}. \tag{7.14}$$

It may be verified that $(\mathcal{J}^{(-)})^2 = -\mathbf{1}$. The two GCS commute, i.e. $[\mathcal{J}^{(+)}, \mathcal{J}^{(-)}] = 0$, as expected for a generealized Kähler geometry. Again, the explicit form (6.7) of the matrices is only needed for checking that $(\mathbb{J}^{(-)})^2 = -\mathbf{1}$, where the complex structure is given by the upper left submatrix in (7.14). Note that although the generalized complex structures commute, the complex structures do not, $[\mathbb{J}^{(+)}, \mathbb{J}^{(-)}] \neq 0$.

The product of the two GCS's is

$$\begin{aligned}
-\mathcal{G} &= \mathcal{J}^{(-)}\mathcal{J}^{(+)}\\
&= \begin{pmatrix} iJuq & -2Ju\omega' & 0 & -Ju \\ -2J'u^t\omega & -iJ'u^tp & -J'u^t & 0 \\ 2i(-\omega Juq + pJ'u^t\omega) & 4\omega Ju\omega' + nJ' - pJ'u^tp & ipJ'u^t & 2\omega Ju \\ 4\omega'J'u^t\omega + nJ - qJuq & 2i(-qJu\omega' + \omega'J'u^tp) & 2\omega'J'u^t & -iqJu \end{pmatrix}.
\end{aligned} \tag{7.15}$$

As a final check that we are indeed dealing with a generalized Kähler geometry, we have verified that indeed $\mathcal{G}^2 = 1$ (independent of the explicit form (6.7) of the submatrices).

Note that the form of $\mathcal{J}^{(\pm)}$ depends on the choice of auxiliary fields in the action. Getting from our form of the GCS's to the form given in chapter 6 of [8] involves a redefinition of the auxiliary fields.

8. Discussion

In this paper we have studied the generalized Kähler geometry present in nonlinear sigma models where the basic fields are $\mathcal{N} = (2, 2)$ semi-chiral multiplets. The underlying bihermitiean geometry has noncommuting complex structures which shows that the geometrical situation is different from that in a sigma model with chiral and twisted chiral multiplets where the bihermitian complex structures necessarily commute. The geometry is governed by a potential $K(\mathbb{X}, \bar{\mathbb{X}}, \mathbb{Y}, \bar{\mathbb{Y}})$ and our results should be useful in constructing explicit models (corresponding to choices of K), e.g., generalizing the discussions in [25]. Our models also lend themselves to topological twisting. Presumably the auxiliary fields then have an interpretation as Batalin-Vilkovisky antifields [16].

One open question is whether a nonlinear sigma model with chiral, twisted and semi-(anti)chiral fields covers the most general situation described by Generalized Complex Geometry. In the traditional bi-hermitean setting in, e.g., [23, 24], claims in both directions have been made. We have not proven or disproven this; clearly a geometric understanding of the conditions that allow a description of sigma models using semi-(anti)chiral multiplets would cast light on this issue.

Finally, it should be interesting to study the model discussed when including boundaries. Investigating the possible boundary conditions that preserve some supersymmetry in an open sigma model has proven a very powerful tool in understanding the geometry of the D-brane where it ends [26, 27], and recently it has been shown that these investigations benefit from being formulated in terms of GCG [28].

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